

Decays of degeneracies in \mathcal{PT} -symmetric ring-shaped lattices

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Abstract

Many apparently non-Hermitian ring-shaped discrete lattices share the appeal with their more popular linear predecessors. Irrespectively of the change of the topology, their dynamics controlled by the nearest-neighbor interaction is shown to remain phenomenologically interesting. For the perturbation-caused removals of spectral degeneracy at exceptional points, in particular, alternative scenarios are illustrated via solvable examples.

1 Introduction

The concept of \mathcal{PT} -symmetry [1] is an inspiring source of experimental activities in several branches of physics like optics [2] or solid-state physics [3, 4] or magnetohydrodynamics [5]. The idea itself originates in quantum mechanics. In a way illustrated by numerous toy models [6, 7] the interest in \mathcal{PT} -symmetry has been motivated there by a counterintuitive contrast between the manifestly non-Hermitian form of a given \mathcal{PT} -symmetric interaction Hamiltonian H and the reality of the spectrum of the related energies inside a certain domain \mathcal{D} of optional parameters.

The mathematical key to the puzzling existence of a non-empty domain \mathcal{D} of parameters giving real spectra has been found in Ref. [8]. The authors emphasized there the compatibility of the non-Hermiticity of H in a given Hilbert space (say, in $\mathcal{H}^{(friendly)} := L^2(\mathbb{R})$ where $H \neq H^\dagger$) with the *simultaneous* Hermiticity of the same operator in another Hilbert space. In particular, the new space $\mathcal{H}^{(sophisticated)}$ may be allowed to coincide with $\mathcal{H}^{(friendly)}$ as a vector space, being just endowed with another, nontrivial, *ad hoc* inner product.

The main phenomenological appeal of such a situation lies, paradoxically, in the possibility of a *loss* of the reality of the spectrum. This happens, naturally, at the boundary $\partial\mathcal{D}$, i.e., at a certain value of parameter(s) called, according to Kato [9], exceptional point(s) (EP). They specify, in effect, the strength of a critical non-Hermiticity at which the *ad hoc* inner product ceases to exist (cf. [10] and [11])

One of the weak points of the recipe (which the authors of Ref. [8] could have called “quasi-Hermitian quantum mechanics”, QHQM) lies in the ambiguity of the definition of the *ad hoc* inner product mediated by the so called metric operator $\Theta = \Theta^\dagger > 0$,

$$\langle\phi|\psi\rangle^{(S)} := \langle\phi|\Theta|\psi\rangle^{(F)}. \quad (1)$$

A way has been found in the so called \mathcal{PT} -symmetric quantum mechanics (PTSQM) as reviewed in Ref. [1]. The essence of PTSQM lies in the introduction of an additional postulate $\mathcal{P}TH = H\mathcal{P}T$ (called \mathcal{PT} -symmetry of the Hamiltonian for a suitable operator \mathcal{PT}) and in its combination with another, *ad hoc* requirement of factorization $\Theta^{(PT)} = \mathcal{P}\mathcal{C}$ where operator \mathcal{C} is a charge (cf. Ref. [1] for more details). The nontriviality of the *additional* requirement of \mathcal{PT} -symmetry proved more than compensated by its heuristic efficiency, with \mathcal{P} chosen, most often, as the operator of parity.

A disappointing failure of the extension of the PTSQM formalism to the dynamical regime of scattering [12] has been discussed in Ref. [13]. The obstacle has been found removable via a return to the QHQM recipe, with the emphasis shifted from the charge in $\Theta^{(PT)}$ to an alternative treatment of the ambiguity of the metric.

The perceivable increase of the technical difficulty of the required explicit construction of the metrics $\Theta \neq \Theta^{(PT)}$ has been softened by the discretiza-

tion of the coordinates, $x \rightarrow x_n$ (cf. also [14]). The common differential toy-model Hamiltonians $H = -\Delta + V(\vec{x})$ were replaced by their difference-operator analogues. Typically, the role of the kinetic energy $-\Delta$ was taken by a tridiagonal matrix (i.e., say, by the well known Runge-Kutta approximation of the Laplacean). Simultaneously, the diagonal potential-energy matrix $V(x_n)$ was generalized into a tridiagonal matrix which was not necessarily Hermitian in $\mathcal{H}^{(friendly)}$.

The resulting N by N Hamiltonians of the generic form

$$\hat{H}^{(N)} = \begin{bmatrix} a_1 & c_1 & 0 & \dots & 0 & 0 \\ b_2 & a_2 & c_2 & 0 & \dots & 0 \\ 0 & b_3 & a_3 & c_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_{N-1} & a_{N-1} & c_{N-1} \\ 0 & 0 & \dots & 0 & b_N & a_N \end{bmatrix} \quad (2)$$

may be interpreted as representing an open-end N -site nearest-neighbor-interaction quantum lattice [3].

In what follows we intend to complement the existing studies of various open-end versions of the N -site quantum lattice (2) (cf., e.g., Refs. [15] or [16], with further references) by the next-step study of its generalization

$$\hat{H}^{(N)} = \begin{bmatrix} a_1 & c_1 & 0 & \dots & 0 & -c_N \\ -c_1 & a_2 & c_2 & 0 & \dots & 0 \\ 0 & -c_2 & a_3 & c_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -c_{N-2} & a_{N-1} & c_{N-1} \\ c_N & 0 & \dots & 0 & -c_{N-1} & a_N \end{bmatrix}, \quad N = 2J. \quad (3)$$

This Hamiltonian matrix may be read as representing a circular, ring-shaped discrete lattice (cf. its graphical representation in Figs. 1 and 2).

The motivation of such a project is twofold. Firstly, the introduction of the “anomalous” matrix elements c_N enables us to study the spectral consequences of the tunable input interaction of a long-range character. Section 2 will pay particular attention to the effects of this long-range perturbation on a *maximally degenerate* EP singularity at $N = 6$. We shall sample alternative scenarios of the removal of this degeneracy.

Secondly, we feel motivated by the simplicity-preserving character of the transition from Eq. (2) to Eq. (3). In section 3 a few exactly solvable benchmark models will be presented, demonstrating an enrichment of the variability of the spectrum in non-perturbative regime. Our models will exhibit *multiple complexifications* of the energies at the EP boundary $\partial\mathcal{D}$.

Our observations and proposals will be finally summarized in section 4.

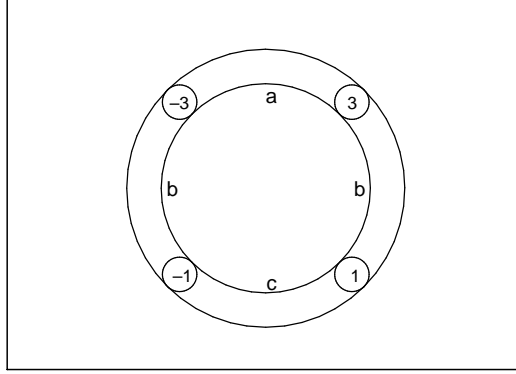


Figure 1: A graphical representation of a sample $N = 4$ lattice (3) with $a_4 = -a_1 = 3$ and $a_3 = -a_2 = 1$ while $a = c_4$, $b = c_3 = c_1$ and $c = c_2$.

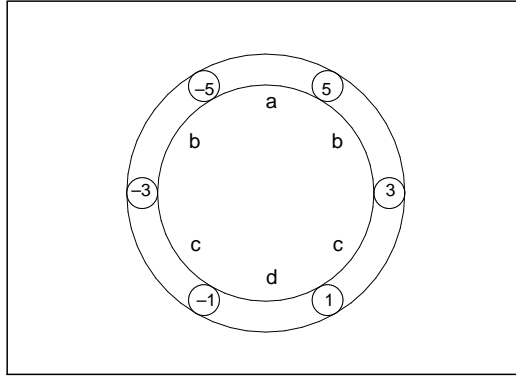


Figure 2: A graphical representation of a sample $N = 6$ lattice (3) with $a_6 = -a_1 = 5$, $a_5 = -a_2 = 3$ and $a_4 = -a_3 = 1$ while $c_6 = a$, $c_5 = c_1 = b$, $c_4 = c_2 = c$ and $c_3 = d$.

2 The decays of a multiple degeneracy

In Ref. [15] we proposed a family of multi-parametric N -dimensional matrices (2) for which one is able to construct certain parts of the boundary $\partial\mathcal{D}$ in closed form, non-numerically. In an $N = 6$ illustration of such a toy-model scenario let us recall the multiple-degeneracy-generating Hamiltonian matrix

$$\begin{aligned}
H_{(MDG)} = & \\
= & \begin{bmatrix} -5 & \sqrt{5-5t} & 0 & 0 & 0 & 0 \\ -\sqrt{5-5t} & -3 & 2\sqrt{2-2t} & 0 & 0 & 0 \\ 0 & -2\sqrt{2-2t} & -1 & 3\sqrt{1-t} & 0 & 0 \\ 0 & 0 & -3\sqrt{1-t} & 1 & 2\sqrt{2-2t} & 0 \\ 0 & 0 & 0 & -2\sqrt{2-2t} & 3 & \sqrt{5-5t} \\ 0 & 0 & 0 & 0 & -\sqrt{5-5t} & 5 \end{bmatrix}.
\end{aligned}
\tag{4}$$

The corresponding spectrum of energies stays unobservable (complex) at the negative “times” $t < 0$ while becoming, suddenly, completely degenerate at $t = 0$ and strictly real and fully non-degenerate at all the positive $t > 0$. This fine-tuned multiple-degeneracy behavior of the spectrum (sampled here in Fig. 3) may immediately be extended to any dimension N [17].

We shall check what may happen when the linear chain of Eq. (2) is being replaced by the ring-shaped chain represented by Eq. (3).

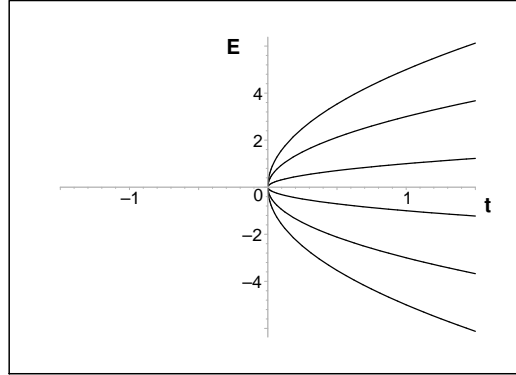


Figure 3: The onset of the reality of the spectrum of energies of the six-site lattice (4) at the Kato’s degenerate exceptional point $t = 0$ (the open-end dynamical regime).

2.1 \mathcal{PT} –symmetry in lattices

It has recently been hinted in the literature that the open-end non-Hermitian quantum lattices are exceptional “since periodic boundary conditions are incompatible with the \mathcal{PT} –symmetry” [3]. For several reasons, such a comment attracted our attention. First of all, we felt puzzled by the potential physical consequences of such a statement since it contradicted our older empirical experience with the existence of strong parallels between the open- and coupled-end quantum lattices [18].

During our subsequent analysis of the problem we imagined that its very core is entirely artificial. The source of the misunderstanding may simply

be traced back to certain ambiguity of the current terminology. In an introductory remark let us, therefore, recall a few relevant definitions while restricting our present attention, for the sake of simplicity, just to Hamiltonians (2) or (3) with *real* matrix elements.

In the first step let us slightly simplify the complex-Hamiltonian conventions as accepted in Ref. [3]. Thus, our present interpretation of the \mathcal{PT} -symmetry property $H\mathcal{PT} = \mathcal{P}TH$ of the real Hamiltonians will employ the time-reversal \mathcal{T} represented just by the operator of matrix transposition. Secondly, the specific choice of the parity operator

$$\mathcal{P} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & \mp 1 \end{bmatrix} \quad (5)$$

will be assumed fixed in advance. Thirdly, let us clearly distinguish between the concepts of the so called unbroken and broken \mathcal{PT} -symmetry where, by definition, the whole spectrum of H is real or not real, respectively.

In a more explicit concise explanation involving, for the sake of brevity, just the non-Hermitian \mathcal{PT} -symmetric Hamiltonians (i.e., matrices $H \neq H^\dagger$ such that $H^\dagger \mathcal{P} = \mathcal{P}H$) with non-degenerate (though, in general, complex) spectra, one must distinguish between the right-eigenvector Schrödinger equation $H|R_n\rangle = E_n|R_n\rangle$ and the left-eigenvector Schrödinger equation $\langle L_n|H = \langle L_n|E_n$ (or, equivalently, $K|L_n\rangle = F_n|L_n\rangle$ where $K \equiv H^\dagger \neq H$ and $F_n \equiv E_n^*$).

In this notation one easily separates the real-energy case “A” (in which $F_n = E_n$) and the complex-energy case “B” (in which $F_n = E_n^* = E_m$ at a subscript $m = m(n) \neq n$). As long as we can always write $\mathcal{P}H|R_n\rangle = H^\dagger(\mathcal{P}|R_n\rangle) = E_n(\mathcal{P}|R_n\rangle)$ in both of these cases, it is easy to conclude that

$$\mathcal{P}|R_n\rangle = \text{const} |L_n\rangle \quad \text{iff} \quad E_n = E_n^*. \quad (6)$$

In the major part of the current literature on \mathcal{PT} -symmetric Hamiltonians, the validity of proportionality (6) between the left and right eigenvectors *at all* $n = 1, 2, \dots, N$ is, conveniently, called the unbroken \mathcal{PT} -symmetry of the quantum system in question. Thus, one must be a bit careful when reading Ref. [3] where the models with unbroken \mathcal{PT} -symmetry are called “models in \mathcal{PT} -symmetric phase”.

This being explained, we believe that there is no true reason for taking the circular lattices (i.e., in our case, systems with Hamiltonians (3) mimicking the periodic boundary conditions and representing the circular lattices of the shape sampled in Fig. 2) as a perceivably more complicated option. We might even conjecture that as long as the circular shape of the lattice

may be perceived as an elementary exemplification of a topologically non-trivial quantum graph of a non-tree shape, the presence of the end-point bonds might be interpreted, in the spirit of Ref. [19], as a hidden source of potentially interesting anomalies in the spectrum.

We shall restrict our attention to the even- N subset of models (3). The main reason is that under this restriction our Hamiltonians will exhibit more parallels with their differential-equation predecessors. In particular, we shall always employ just the manifestly coupling-independent operator \mathcal{P} of Eq. (5) which strongly resembles the standard parity with its equal number of positive and negative eigenvalues.

2.2 A destabilization via a coupling between ends

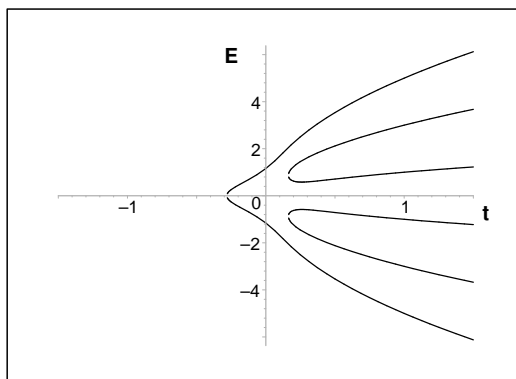


Figure 4: The change of the spectrum of Fig. 3 after the transition to the ring-shaped lattice of Fig. 2 in the weak-coupling dynamical regime.

For a sampling of the effects of the periodicity-simulating lattice-ends couplings $\pm c_N$ let us return to the six-site open-end-lattice spectrum of Fig. 3 and let us treat the bonding matrix elements $\pm c_N$ as a small perturbation to this unperturbed form of the \mathcal{PT} -symmetric Hamiltonian.

Let us consider the first sample of such a perturbed Hamiltonian $H_1^{(6)}(t) = H_{(MDG)} + W_1$, with the perturbation specified as follows,

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -w \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad w = \frac{1}{100}\sqrt{1-t}.$$

The resulting modified t -dependence of the spectrum is displayed in Fig. 4. We see that the original collapse of the whole spectrum gets split. A twin

partial collapse is shifted to the right (i.e., to $t \rightarrow 0.159^+$) while the ultimate complete complexification moved to the left (i.e., to negative $t \rightarrow -0.2818^+$).

2.3 Competing agents of destabilization

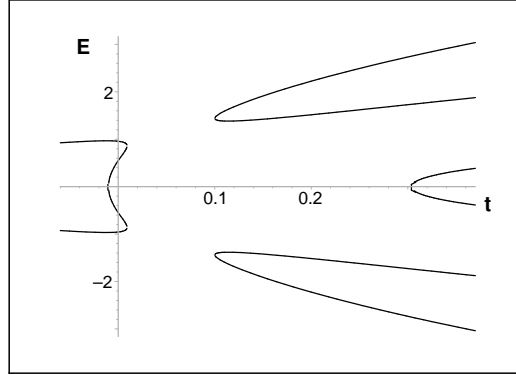


Figure 5: The energies of the six-site lattice (7) (Fig. 2) with a stronger bond between the endpoints and with an enhanced central attraction.

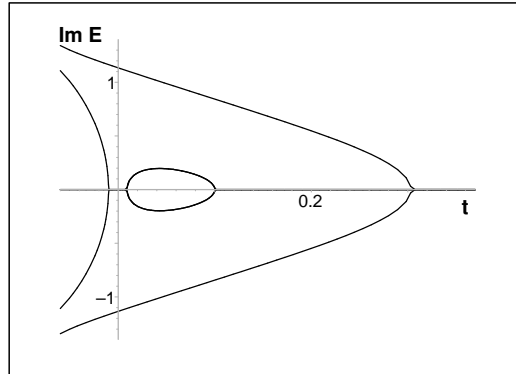


Figure 6: A pendant of Fig. 5 – the imaginary parts of the energies.

From the observations made in paragraph 2.2 one can conclude that both the maximal degeneracy (at $t = 0$) and the Big-Bang-like behavior (at the small $t > 0$) of the spectrum as sampled by Fig. 3 are in fact very sensitive to perturbations. The form of this perturbation is highly relevant for the guarantee of the stability of the perturbed quantum system, i.e., of the reality of its bound-state energies.

The determination of the boundary $\partial\mathcal{D}$ of the domain \mathcal{D} of the parameters which are compatible with the stability is important. Certain qualitative features of this boundary (or “horizon”) may even be guessed *a priori*. Typically, one may expect that the system gets less stable, i.e., the size of the domain \mathcal{D} will shrink in the strong-coupling regime. In parallel, the destabilizing effect of the growth of the end-bond $c_6 = a$ (cf. Fig. 2) may be compensated

by the decrease (or, alternatively, enhanced by the further increase) of some other non-Hermiticity measure (i.e., for the sake of definiteness, of $c_3 = d$ in Fig. 2).

The consequences are both interesting and hardly predictable. Figures 5 and 6 offer an illustration of the effect. These pictures illustrate a scenario of destabilization in which, in spite of the perceivable growth of a (i.e., in spite of our making the ring better bound), the dominant role is still played by the smaller perturbation of the close-to-critical d .

This guess may be assisted by the toy-model Hamiltonian $H_2^{(6)}(t) = H_{(MDG)} + W_2$ with

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -10w \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 \\ 0 & 0 & -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 10w & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad w = \frac{1}{100}\sqrt{1-t}. \quad (7)$$

The mechanism of the dominance of the enhancement of the central coupling is demonstrated by Figs. 5 and 6. We see there the remarkable pattern of complexification in which the rightmost exceptional point $t_+ \approx 0.3$ of the loss of the crypto-Hermiticity (i.e., of the reality of the whole spectrum) is determined by d while the next coordinate $t_- \approx 0.1$ of the remaining two mergers and subsequent complexifications already reflects the combined effect of the whole perturbation.

A qualitative novelty may be seen in the re-emergence of a small island of the reality of as many as four central energies in a very small vicinity of $t = 0$. This phenomenon is a close analogue of the similar spectral “reality-island” anomaly encountered, in Ref. [19], in the strong-coupling regime of another topologically nontrivial model.

3 Decays of separate degeneracies

The explicit evaluation of the periodic-lattice spectra remains a more or less purely numerical problem in the tight-binding regime, especially at the larger N . One still encounters exactly solvable secular equations at the smallest even $N = 2J$. *In extremis*, many generic features of the periodic-lattice spectra may be understood even via their first nontrivial four-site-lattice realization.

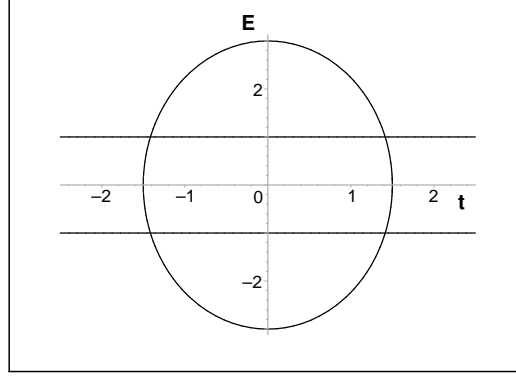


Figure 7: The spectrum of energies of the periodic equal-coupling four-site-lattice Hamiltonian (8).

3.1 An exactly solvable model

Once we pick up the following one-parametric four-site toy model with equal couplings,

$$H_{(EC)} = \begin{bmatrix} -3 & t & 0 & -t \\ -t & -1 & t & 0 \\ 0 & -t & 1 & t \\ t & 0 & -t & 3 \end{bmatrix} \quad (8)$$

we reveal that the whole spectrum remains real up to the strong-coupling dynamical regime, i.e., even beyond $|t| = 1$ (cf. the graphical representation of this spectrum in Fig. 7). For our present purposes it is also useful that the spectrum is in fact available in closed form,

$$E_0 = -\sqrt{9 - 4t^2}, \quad E_1 = -1, \quad E_2 = 1, \quad E_3 = \sqrt{9 - 4t^2}. \quad (9)$$

One localizes, precisely, the unavoided-crossing points $t = t_{\pm}^{(UC)} = \pm\sqrt{2}$ as well as the points of the ultimate complexification $t = t_{\pm}^{(C)} = \pm 3/2$. All of these four values are the exceptional points in the Kato's sense [9]. In order to see this, one may recall Eq. (9) and pick up, say, eigenvalues E_2 and E_3 of our Hamiltonian (8). Then, the two respective eigenvectors, viz., the four-component vector $\vec{\psi}_2 = [0, t, 2, t]$ and its partner $\vec{\psi}_3$ with components

$$\left[t^2 - 2, \left(\sqrt{9 - 4t^2} + 1 \right) t/2, 3 - t^2 + \sqrt{9 - 4t^2}, \left[\vec{\psi}_3 \right]_4 \right],$$

$$\left[\vec{\psi}_3 \right]_4 = \frac{(4 - t^2) \sqrt{9 - 4t^2} + 12 - 5t^2}{2t}$$

will strictly coincide in either of the limits of $t \rightarrow t_{\pm}^{(UC)}$. The Hamiltonian will only remain diagonalizable (i.e., crypto-Hermitian) inside the three separate intervals of t , viz, inside domain $\mathcal{D}_0^{(H)} = (-\sqrt{2}, \sqrt{2})$ (combining the weak-

and strong-coupling dynamical regimes) or inside $\mathcal{D}_-^{(H)} = (-3/2, -\sqrt{2})$ or $\mathcal{D}_+^{(H)} = (\sqrt{2}, 3/2)$ (= the two perceivably smaller domains of anomalously strong couplings).

Inside the non-anomalous domain $\mathcal{D} = (-\sqrt{2}, \sqrt{2})$ of t our choice of the first nontrivial $N = 4$ enables us to find a *complete* family of all of the candidates for the metric in principle. A sample of the necessary linear algebra reconstructing, basically, the metric from the relation

$$H^\dagger = \Theta H \Theta^{-1} \quad (10)$$

may be found, say, in Ref. [13]. An entirely exhaustive explicit construction of the metrics Θ has been performed there for a certain extremely elementary one-parametric model. Here, we shall only pick up a single, characteristic solution

$$\Theta = \begin{bmatrix} 3+t^2 & -3t & t^2 & t \\ -3t & 3+t^2 & -3t & t^2 \\ t^2 & -3t & 3+t^2 & -3t \\ t & t^2 & -3t & 3+t^2 \end{bmatrix} \quad (11)$$

as a candidate for the metric in $\mathcal{H}^{(sophisticated)}$. In order to confirm its eligibility we must prove that it is positive definite. Such a proof is easy since the two eigenvalues of our candidate matrix happen to possess the elementary form

$$3 + t^2 + t \pm \sqrt{13t^2 + t^4 + 6t^3}.$$

The other two eigenvalues are obtained by the replacement $t \rightarrow -t$. This implies the positivity of our matrix (11) inside the interval of

$$t \in (-\sqrt{3/2}, \sqrt{3/2}) := \mathcal{D}_\Theta \approx (-1.225, 1.225).$$

We see that our choice of special metric is satisfactory since this interval covers more than 86 % of the whole crypto-Hermiticity domain of the Hamiltonian itself.

3.2 The strengthened bond

Let us now replace the fine-tuned one-parametric four-site model (8) by its perturbation

$$H = H_{(EC)} + \begin{bmatrix} 0 & 0 & 0 & -t/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t/2 & 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

Figure 8 illustrates the changes. The decay of the two original point-like unavoided crossings at $t_\pm^{(UC)}$ spreads over the respective two small non-empty

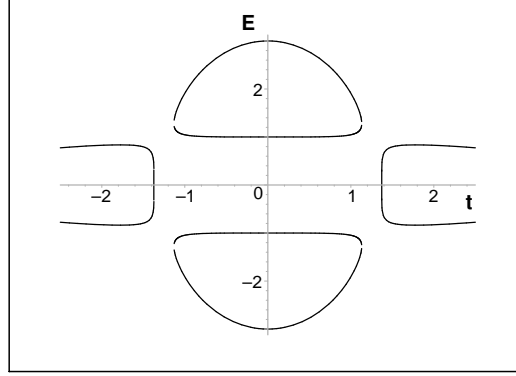


Figure 8: The change of the spectrum of Fig. 7 caused by the 50% enhancement of the value of the periodicity-guaranteeing bond.

intervals of t in which all of the four eigenenergies become complex, acquiring a non-vanishing imaginary part.

A more detailed inspection of Fig. 8 reveals that the central domain shrinks a bit in comparison, $\mathcal{D}_0^{(H)} \approx (-1.13137, 1.13137)$. The ultimate onset of the large- t survival of the two real energies moves from the points $t_{\pm}^{(C)} = 3/2$ slightly closer to the origin.

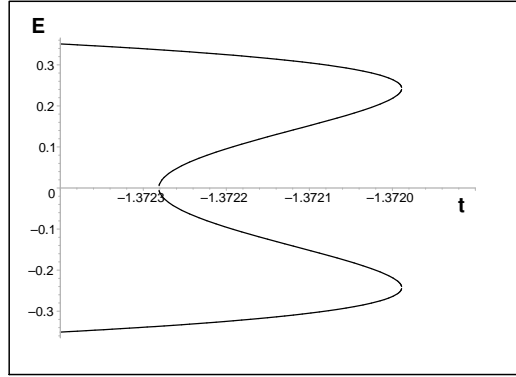


Figure 9: Same as Fig. 8, in a more detailed blow up.

Figure 8 seems to indicate the complete disappearance of the separate anomalous domains $\mathcal{D}_{\pm}^{(H)}$. The magnified version of the same picture (cf. Fig. 9) reveals that it is not so. The two very small but still non-empty intervals $\mathcal{D}_+^{(H)} \approx (1.37199, 1.37228)$ and $\mathcal{D}_-^{(H)} \approx (-1.37228, -1.37199)$ survive and yield the whole quadruplet of the energies purely real again, in a truly fine-tuned manner.

A challenging task emerges in connection with an appropriate modification of the metric. In place of Eq. (11) the present would-be positive-definite

solution of Eq. (10) reads

$$\begin{bmatrix} 3+t^2 & \frac{(3+t^2)t(13t^2-96)}{17t^2+96} & 24\frac{(3+t^2)t^2}{17t^2+96} & 1/2\frac{(3+t^2)t(t^2+96)}{17t^2+96} \\ \frac{(3+t^2)t(13t^2-96)}{17t^2+96} & 3+t^2 & \frac{(3+t^2)t(7t^2-96)}{17t^2+96} & 24\frac{(3+t^2)t^2}{17t^2+96} \\ 24\frac{(3+t^2)t^2}{17t^2+96} & \frac{(3+t^2)t(7t^2-96)}{17t^2+96} & 3+t^2 & \frac{(3+t^2)t(13t^2-96)}{17t^2+96} \\ 1/2\frac{(3+t^2)t(t^2+96)}{17t^2+96} & 24\frac{(3+t^2)t^2}{17t^2+96} & \frac{(3+t^2)t(13t^2-96)}{17t^2+96} & 3+t^2 \end{bmatrix}.$$

In spite of its perceivably more complicated structure this matrix may still be shown, by the same techniques as above, to be safely positive definite (i.e., to become eligible as a metric) for

$$t \in \mathcal{D}_\Theta \approx (-1.082854389, 1.082854389)$$

i.e., in a slightly diminished range of the “time” parameter.

3.3 A recoupled regime

In a way complementing Eq. (12) let us consider

$$H = H_{(EC)} + \begin{bmatrix} 0 & 0 & 0 & 3t/4 \\ 0 & 0 & t/3 & 0 \\ 0 & -t/3 & 0 & 0 \\ -3t/4 & 0 & 0 & 0 \end{bmatrix}. \quad (13)$$

The fine-tuned model (8) looks now perturbed in an opposite direction. The perceivable weakening of the matrix element in the corner (i.e., of the periodicity-guaranteeing bond) enhances the parallels with the open-end systems. By itself, this change should lead to a completely complex spectrum at the larger $|t|$ s.

This expectation is confirmed by Fig. 10. The picture shows that the choice of the stronger central attraction in Eq. (13) works in the fragility-enhancing direction, diminishing the central crypto-Hermiticity domain,

$$\mathcal{D}_0^{(H)} := \left(\frac{3\sqrt{97}-45}{16}, \frac{45-3\sqrt{97}}{16} \right) \approx (-0.9658391622, 0.9658391622).$$

The presence of the inflexions in the outer energy loop finds its origin in the highly unstable unavoided-crossing points of model (8). The relevance of the inflexion points as emphasized in Ref. [3] might be recalled.

The construction of the metric preserves the algebraic structure shown above. From an updated formula (using just different numerical coefficients)

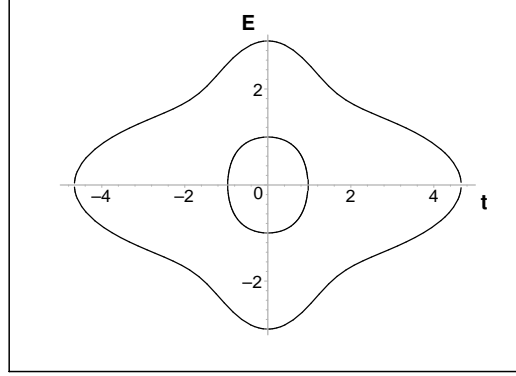


Figure 10: The thorough deformation of the spectrum of Fig. 7 caused by the recompensation plus weakening of the periodicity-guaranteeing bond.

we derive the boundary points of the interval \mathcal{D}_Θ . The definition of these boundaries is provided by the minimal root of the expression

$$\left(-\frac{1235}{8}t^3 + 631t^2 - 720t + 1152 - 1/24\sqrt{V}\right)(3+t^2)(631t^2 + 1152)^{-1},$$

$V = 2772145t^6 - 50595840t^5 + 348491520t^4 - 955514880t^3 + 1147944960t^2$
giving

$$t \in \mathcal{D}_\Theta \approx (-0.9658391622, 0.9658391622).$$

The size of this interval of the positivity of Θ is again comparable with the size of the interval of the crypto-Hermiticity of the Hamiltonian.

4 Outlook

The recent growth of interest in \mathcal{PT} -symmetric quantum lattices offers a natural motivation of the transition to the loop-shaped lattices. We found that a core of the consistency of such a transition (which could suffer from its potential “fragility” in general [20]) is similar to the suppression of the fragility in open-ends models.

Via a thorough description of a few not too complicated examples we illustrated that in *both* the open-end and matched-ends models, the stability and the robust nature (i.e., non-fragile nature) of the models results from the absence of the degeneracy of spectra of the zero-coupling versions of the Hamiltonians in question.

The role of the matching matrix elements of the Hamiltonian remains, in the phenomenological perspective, slightly counterintuitive. Still, the decisive conceptual parallels between periodic lattices and the mathematically more friendly open-end lattices were noticed. They involve not only the encouraging emergence of the parallel structures in the spectra of energies but

also in the shapes of the domains of the reality of the energies or of the positivity of the matrix candidates for the metrics.

An important merit of our specific models may be seen in the feasibility of calculations. This resulted from the preservation of multiple parallels between our present matched-ends models and their open-ends predecessors (let us mention just the up-down symmetry, equidistance of the unperturbed spectra or the reality of the interaction matrices). Nevertheless, even beyond the level of the low-dimensional solvable examples the more general questions of consistency of the underlying quantum theory were addressed. Our constructive study of the chains defined along discrete loops appeared more friendly than expected.

Our detailed analyses covered the extensive dynamical domain, far beyond the mere weak-coupling subdomain. Our periodicity-simulating bonds proved connected to the emergence of unexpected spectral phenomena (like the strong-coupling-related islands of stability) which will certainly deserve a further study in the future.

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